

Vertex (Lie) algebras in higher dimensions

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Abstract. Vertex algebras provide an axiomatic algebraic description of the operator product expansion (OPE) of chiral fields in 2-dimensional conformal field theory. Vertex Lie algebras (= Lie conformal algebras) encode the singular part of the OPE, or, equivalently, the commutators of chiral fields. We discuss generalizations of vertex algebras and vertex Lie algebras, which are relevant for higher-dimensional quantum field theory.

1 Vertex algebras and Lie conformal algebras

In the theory of *vertex algebras* [1, 2, 3], the (quantum) *fields* are linear maps from V to $V[[z]][z^{-1}]$, where z is a formal variable. They can be viewed as formal series $a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$ with $a_{(n)} \in \text{End}V$ such that $a_{(n)}b = 0$ for n large enough. Let $\text{Res } a(z) = a_{(0)}$; then the *modes* of $a(z)$ are given by $a_{(n)} = \text{Res } z^n a(z)$. The *locality* condition for two fields

$$(z - w)^{N_{ab}} [a(z), b(w)] = 0, \quad N_{ab} \in \mathbb{N}$$

is equivalent to the *commutator formula*

$$[a(z), b(w)] = \sum_{j=0}^{N_{ab}-1} c_j(w) \partial_w^j \delta(z - w) / j!$$

for some new fields $c_j(w)$, where $\delta(z - w)$ is the formal delta-function (see [2]). The *operator product expansion* (OPE) can be written symbolically

$$a(z)b(w) = \sum_{j \in \mathbb{Z}} c_j(w) (z - w)^{-j-1}$$

(see [2] for a rigorous treatment). The new field c_j is called the *j-th product* of a, b and is denoted $a_{(j)}b$. The *Wick product* (= normally ordered product)

coincides with $a_{(-1)}b$. The j -th products satisfy the *Borcherds identity* [2, Eq (4.8.3)]. Finally, recall that every vertex algebra V is endowed with a *translation operator* $T \in \text{End } V$ satisfying $[T, a(z)] = \partial_z a(z) = (Ta)(z)$.

The commutator of two fields is encoded by the singular part of their OPE and is uniquely determined by their j -th products for $j \geq 0$. The λ -bracket

$$[a_\lambda b] = \text{Res}_z e^{z\lambda} a(z)b = \sum_{j=0}^{N_{ab}-1} \lambda^j a_{(j)}b/j!$$

satisfies the axioms of a *Lie conformal algebra* introduced by Kac [2] (also known as a *vertex Lie algebra*; cf. [3, 4]). This is a $\mathbb{C}[T]$ -module R with a \mathbb{C} -linear map $R \otimes R \rightarrow R[\lambda]$ satisfying:

$$\begin{aligned} \text{sesquilinearity} \quad & [(Ta)_\lambda b] = -\lambda [a_\lambda b], \quad [a_\lambda (Tb)] = (T + \lambda)[a_\lambda b], \\ \text{skewsymmetry} \quad & [a_\lambda b] = -[b_{-\lambda} a], \\ \text{Jacobi identity} \quad & [a_\lambda [b_\mu c]] - [b_\mu [a_\lambda c]] = [[a_\lambda b]_{\lambda+\mu} c]. \end{aligned}$$

The relationship between Lie conformal algebras and vertex algebras is somewhat similar to the one between Lie algebras and their universal enveloping associative algebras (see [2, 5]). In [5] we gave a definition of the notion of a vertex algebra as a Lie conformal algebra equipped with one additional product, which becomes the Wick product. Lie conformal (super)algebras were classified in [6]; their representation theory and cohomology theory were developed in [7].

2 Lie pseudoalgebras

Lie pseudoalgebras are “multi-dimensional” generalizations of Lie conformal algebras [8]. In the definition from the previous section, we consider λ , μ , etc., as D -dimensional vector variables, we replace T by $\mathbf{T} = (T_1, \dots, T_D)$, and $\mathbb{C}[T]$ by $\mathbb{C}[\mathbf{T}] \equiv \mathbb{C}[T_1, \dots, T_D]$. For $D = 0$ we let $\mathbb{C}[\mathbf{T}] \equiv \mathbb{C}$; then a Lie pseudoalgebra is just a usual Lie algebra.

Examples of Lie pseudoalgebras:

1. Cur $\mathfrak{g} = \mathbb{C}[\mathbf{T}] \otimes \mathfrak{g}$, $[a_\lambda b] = [a, b]$ for $a, b \in \mathfrak{g}$, where \mathfrak{g} is a Lie algebra.
2. $W(D) = \mathbb{C}[\mathbf{T}]L^1 \oplus \dots \oplus \mathbb{C}[\mathbf{T}]L^D$, $[L^\alpha_\lambda L^\beta] = (T_\alpha + \lambda_\alpha)L^\beta + \lambda_\beta L^\alpha$.
3. $S(D, \chi) = \{\sum P_\alpha(\mathbf{T})L^\alpha \mid \sum (\partial_{T_\alpha} + \chi_\alpha)P_\alpha(\mathbf{T}) = 0\} \subset W(D)$, $\chi \in \mathbb{C}^D$.
4. $H(D) = \mathbb{C}[\mathbf{T}]L$, D – even, $[L_\lambda L] = \sum_{\alpha=1}^{D/2} (\lambda_\alpha T_{\alpha+\frac{D}{2}} - \lambda_{\alpha+\frac{D}{2}} T_\alpha)L$.

More generally than in Example 1, a *current pseudoalgebra* over a Lie pseudoalgebra R is defined by tensoring R with $\mathbb{C}[T_1, \dots, T_{D'}]$ over $\mathbb{C}[\mathbf{T}]$ and keeping the same λ -bracket for elements of R , where $D' > D$.

Theorem ([8]). *Every simple Lie pseudoalgebra, which is finitely generated over $\mathbb{C}[T]$, is a current pseudoalgebra over one of the above.*

In fact, in [8] we introduced and studied a more general notion of a Lie pseudoalgebra, in which $\mathbb{C}[T]$ is replaced by the universal enveloping algebra of a Lie algebra of symmetries. Lie pseudoalgebras are closely related to the Lie–Cartan algebras of vector fields: $W_D = \text{Der } \mathbb{C}[[x_1, \dots, x_D]]$ (Witt algebra), $S_D \subset W_D$ (divergence zero), $H_D \subset W_D$ (hamiltonian), $K_D \subset W_D$ (contact). Lie pseudoalgebras are also related to Ritt’s differential Lie algebras, linear Poisson brackets in the calculus of variations, classical Yang–Baxter equation, and Gelfand–Fuchs cohomology (see [8]). The irreducible representations of the above Lie pseudoalgebras were classified in [9].

3 Vertex algebras and vertex Lie algebras in higher dimensions

“Multi-dimensional” generalizations of vertex algebras were considered in [10, 11]. The ones introduced by Nikolov in [11] arose naturally within a one-to-one correspondence with axiomatic *quantum field theory* models satisfying the additional symmetry condition of *global conformal invariance* [12]. The theory of these vertex algebras was developed in [11, 13]. The main difference with the usual vertex algebras discussed in Section 1 (which correspond to $D = 1$) is that now $z = (z^1, \dots, z^D)$ is a vector variable. A *field* on V is defined as a linear map from V to $V[[z]][1/z^2]$, where $z^2 \equiv zz = z^1z^1 + \dots + z^Dz^D$ (so the singularities of fields are supported on the light-cone). Fields have a *mode* expansion

$$a(z) = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}_{\geq 0}} \sum_{\sigma=1}^{\mathfrak{h}_m} a_{\{n,m,\sigma\}} (z^2)^n h_{m,\sigma}(z), \quad a_{\{n,m,\sigma\}} \in \text{End } V,$$

where $\{h_{m,\sigma}(z)\}_{\sigma=1, \dots, \mathfrak{h}_m}$ is a basis of the space of harmonic homogeneous polynomials of degree m . Fields have the property

$$(z^2)^{N_{ab}} a(z)b \in V[[z]] \equiv V[[z^1, \dots, z^D]], \quad N_{ab} \in \mathbb{N},$$

and the *locality* condition is now

$$((z-w)^2)^{N_{ab}} [a(z), b(w)] = 0.$$

A vertex algebra V is now endowed with D commuting *translation operators* $T_1, \dots, T_D \in \text{End } V$ satisfying $[T_\alpha, a(z)] = \partial_{z^\alpha} a(z) = (T_\alpha a)(z)$. In [13] we defined the *residue* by $\text{Res } a(z) = a_{\{-\frac{D}{2}, 0, 1\}}$ if $h_{0,1}(z) \equiv 1$; then it is translation

invariant and the modes of fields can be obtained as residues. Introduce the notation $\iota_{z,w}F(z-w)$ for the formal Taylor expansion $e^{-w\partial_z}F(z)$.

Theorem ([13]). *In any vertex algebra we have the Borcherds identity*

$$\begin{aligned} & a(z)b(w)c\iota_{z,w}F(z,w) - b(w)a(z)c\iota_{w,z}F(z,w) \\ &= (z^2)^{-L}\left[((u+z-w)^2)^L(\iota_{z,w} - \iota_{w,z})(a(z-w)b)(u)cF(z,w)\right]_{u=w} \end{aligned}$$

for $L \geq N_{ac}$ and $F(z,w) \in \mathbb{C}[z,w,1/z^2,1/w^2,1/(z-w)^2]$.

For $F(z,w) = 1$ the above identity reduces to a *commutator formula*. Because of $\iota_{z,w} - \iota_{w,z}$ only the *singular part* of $a(z-w)$ contributes, where

$$a(z)_{s.p.} = \sum_{n \in \mathbb{Z}_{<0}} \sum_{m \in \mathbb{Z}_{\geq 0}} \sum_{\sigma=1}^{\mathfrak{h}_m} a_{\{n,m,\sigma\}} (z^2)^n h_{m,\sigma}(z).$$

The singular parts of fields satisfy the *Jacobi identity* [13]

$$\begin{aligned} & [a(z)_{s.p.}, b(w)_{s.p.}]c \\ &= \left(\left[(z^2)^{-L}((u+z-w)^2)^L(\iota_{z,w} - \iota_{w,z})(a(z-w)_{s.p.}b)(u)_{s.p.}c \right]_{u=w} \right)_{s.p.}. \end{aligned}$$

We also have *translation invariance* $[T_\alpha, a(z)_{s.p.}] = \partial_z^\alpha a(z)_{s.p.} = (T_\alpha a)(z)_{s.p.}$ and *skewsymmetry* $a(z)_{s.p.}b = (e^{zT}(b(-z)a))_{s.p.}$. The above three axioms define the notion of a *vertex Lie algebra* in higher dimensions [13]. From a vertex Lie algebra one gets a vertex algebra by adding the *Wick product* $a_{\{0,0,1\}}b$, similarly to the construction of [5] for $D = 1$.

Example. The modes of the *real bilocal field* $V(z,w)$ from [14] obey the Lie algebra $\mathfrak{sp}(\infty, \mathbb{R})$; hence it gives rise to a vertex Lie algebra. The conformal Lie algebra $\mathfrak{so}(D,2)$ can be embedded in a suitably completed and centrally extended $\mathfrak{sp}(\infty, \mathbb{R})$. The corresponding vertex algebra is *conformal* and *unitary*. Its unitary positive-energy representations were determined in [14].

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